

# 1. The Relativistic String

All lecture courses on string theory start with a discussion of the point particle. Ours is no exception. We'll take a flying tour through the physics of the relativistic point particle and extract a couple of important lessons that we'll take with us as we move onto string theory.

## 1.1 The Relativistic Point Particle

We want to write down the Lagrangian describing a relativistic particle of mass  $m$ . In anticipation of string theory, we'll consider  $D$ -dimensional Minkowski space  $\mathbf{R}^{1,D-1}$ . Throughout these notes, we work with signature

$$\eta_{\mu\nu} = \text{diag}(-1, +1, +1, \dots, +1)$$

Note that this is the opposite signature to my quantum field theory notes.

If we fix a frame with coordinates  $X^\mu = (t, \vec{x})$  the action is simple:

$$S = -m \int dt \sqrt{1 - \dot{\vec{x}} \cdot \dot{\vec{x}}}. \quad (1.1)$$

To see that this is correct we can compute the momentum  $\vec{p}$ , conjugate to  $\vec{x}$ , and the energy  $E$  which is equal to the Hamiltonian,

$$\vec{p} = \frac{m \dot{\vec{x}}}{\sqrt{1 - \dot{\vec{x}} \cdot \dot{\vec{x}}}} \quad , \quad E = \sqrt{m^2 + \vec{p}^2} \quad ,$$

both of which should be familiar from courses on special relativity.

Although the Lagrangian (1.1) is correct, it's not fully satisfactory. The reason is that time  $t$  and space  $\vec{x}$  play very different roles in this Lagrangian. The position  $\vec{x}$  is a dynamical degree of freedom. In contrast, time  $t$  is merely a parameter providing a label for the position. Yet Lorentz transformations are supposed to mix up  $t$  and  $\vec{x}$  and such symmetries are not completely obvious in (1.1). Can we find a new Lagrangian in which time and space are on equal footing?

One possibility is to treat both time and space as labels. This leads us to the concept of field theory. However, in this course we will be more interested in the other possibility: we will promote time to a dynamical degree of freedom. At first glance, this may appear odd: the number of degrees of freedom is one of the crudest ways we have to characterize a system. We shouldn't be able to add more degrees of freedom

at will without fundamentally changing the system that we're talking about. Another way of saying this is that the particle has the option to move in space, but it doesn't have the option to move in time. It *has* to move in time. So we somehow need a way to promote time to a degree of freedom without it really being a true dynamical degree of freedom! How do we do this? The answer, as we will now show, is gauge symmetry.

Consider the action,

$$S = -m \int d\tau \sqrt{-\dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu}}, \quad (1.2)$$

where  $\mu = 0, \dots, D-1$  and  $\dot{X}^\mu = dX^\mu/d\tau$ . We've introduced a new parameter  $\tau$  which labels the position along the worldline of the particle as shown by the dashed lines in the figure. This action has a simple interpretation: it is just the proper time  $\int ds$  along the worldline.

Naively it looks as if we now have  $D$  physical degrees of freedom rather than  $D-1$  because, as promised, the time direction  $X^0 \equiv t$  is among our dynamical variables:  $X^0 = X^0(\tau)$ . However, this is an illusion. To see why, we need to note that the action (1.2) has a very important property: reparameterization invariance. This means that we can pick a different parameter  $\tilde{\tau}$  on the worldline, related to  $\tau$  by any monotonic function

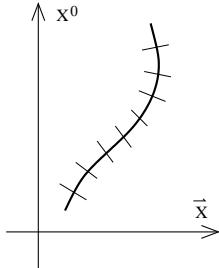
$$\tilde{\tau} = \tilde{\tau}(\tau) .$$

Let's check that the action is invariant under transformations of this type. The integration measure in the action changes as  $d\tau = d\tilde{\tau} |d\tau/d\tilde{\tau}|$ . Meanwhile, the velocities change as  $dX^\mu/d\tau = (dX^\mu/d\tilde{\tau})(d\tilde{\tau}/d\tau)$ . Putting this together, we see that the action can just as well be written in the  $\tilde{\tau}$  reparameterization,

$$S = -m \int d\tilde{\tau} \sqrt{-\frac{dX^\mu}{d\tilde{\tau}} \frac{dX^\nu}{d\tilde{\tau}} \eta_{\mu\nu}} .$$

The upshot of this is that not all  $D$  degrees of freedom  $X^\mu$  are physical. For example, suppose you find a solution to this system, so that you know how  $X^0$  changes with  $\tau$  and how  $X^1$  changes with  $\tau$ , and so on. Not all of that information is meaningful because  $\tau$  itself is not meaningful. In particular, we could use our reparameterization invariance to simply set

$$\tau = X^0(\tau) \equiv t \quad (1.3)$$



**Figure 3:**

If we plug this choice into the action (1.2) then we recover our initial action (1.1). The reparameterization invariance is a *gauge symmetry* of the system. Like all gauge symmetries, it's not really a symmetry at all. Rather, it is a redundancy in our description. In the present case, it means that although we seem to have  $D$  degrees of freedom  $X^\mu$ , one of them is fake.

The fact that one of the degrees of freedom is a fake also shows up if we look at the momenta,

$$p_\mu = \frac{\partial L}{\partial \dot{X}^\mu} = \frac{m \dot{X}^\nu \eta_{\mu\nu}}{\sqrt{-\dot{X}^\lambda \dot{X}^\rho \eta_{\lambda\rho}}} \quad (1.4)$$

These momenta aren't all independent. They satisfy

$$p_\mu p^\mu + m^2 = 0 \quad (1.5)$$

This a constraint on the system. It is, of course, the mass-shell constraint for a relativistic particle of mass  $m$ . From the worldline perspective, it tells us that the particle isn't allowed to sit still in Minkowski space: at the very least, it had better keep moving in a timelike direction with  $(p^0)^2 \geq m^2$ .

One advantage of the action (1.2) is that the Poincaré symmetry of the particle is now manifest, appearing as a global symmetry on the worldline

$$X^\mu \rightarrow \Lambda^\mu_\nu X^\nu + c^\mu \quad (1.6)$$

where  $\Lambda$  is a Lorentz transformation satisfying  $\Lambda^\mu_\nu \eta^{\nu\rho} \Lambda^\sigma_\rho = \eta^{\mu\sigma}$ , while  $c^\mu$  corresponds to a constant translation. We have made all the symmetries manifest at the price of introducing a gauge symmetry into our system. A similar gauge symmetry will arise in the relativistic string, and much of this course will be devoted to understanding its consequences.

### 1.1.1 Quantization

It's a trivial matter to quantize this action. We introduce a wavefunction  $\Psi(X)$ . This satisfies the usual Schrödinger equation,

$$i \frac{\partial \Psi}{\partial \tau} = H \Psi .$$

But, computing the Hamiltonian  $H = \dot{X}^\mu p_\mu - L$ , we find that it vanishes:  $H = 0$ . This shouldn't be surprising. It is simply telling us that the wavefunction doesn't depend on

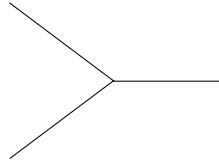
$\tau$ . Since the wavefunction is something physical while, as we have seen,  $\tau$  is not, this is to be expected. Note that this doesn't mean that time has dropped out of the problem. On the contrary, in this relativistic context, time  $X^0$  is an operator, just like the spatial coordinates  $\vec{x}$ . This means that the wavefunction  $\Psi$  is immediately a function of space and time. It is not like a static state in quantum mechanics, but more akin to the fully integrated solution to the non-relativistic Schrödinger equation.

The classical system has a constraint given by (1.5). In the quantum theory, we impose this constraint as an operator equation on the wavefunction, namely  $(p^\mu p_\mu + m^2)\Psi = 0$ . Using the usual representation of the momentum operator  $p_\mu = -i\partial/\partial X^\mu$ , we recognize this constraint as the Klein-Gordon equation

$$\left( -\frac{\partial}{\partial X^\mu} \frac{\partial}{\partial X^\nu} \eta^{\mu\nu} + m^2 \right) \Psi(X) = 0 \quad (1.7)$$

Although this equation is familiar from field theory, it's important to realize that the interpretation is somewhat different. In relativistic field theory, the Klein-Gordon equation is the equation of motion obeyed by a scalar field. In relativistic quantum mechanics, it is the equation obeyed by the wavefunction. In the early days of field theory, the fact that these two equations are the same led people to think one should view the wavefunction as a classical field and quantize it a second time. This isn't correct, but nonetheless the language has stuck and it is common to talk about the point particle perspective as "first quantization" and the field theory perspective as "second quantization".

So far we've considered only a free point particle. How can we introduce interactions into this framework? We would have to first decide which interactions are allowed: perhaps the particle can split into two; perhaps it can fuse with other particles? Obviously, there is a huge range of options for us to choose from. We would then assign amplitudes for these processes to happen. There would be certain restrictions coming from the requirement of unitarity which, among other things, would lead to the necessity of anti-particles. We could draw diagrams associated to the different interactions — an example is given in the figure — and in this manner we would slowly build up the Feynman diagram expansion that is familiar from field theory. In fact, this was pretty much the way Feynman himself approached the topic of QED. However, in practice we rarely construct particle interactions in this way because the field theory framework provides a much better way of looking at things. In contrast, this way of building up interactions is exactly what we will later do for strings.



**Figure 4:**

### 1.1.2 Ein Einbein

There is another action that describes the relativistic point particle. We introduce yet another field on the worldline,  $e(\tau)$ , and write

$$S = \frac{1}{2} \int d\tau \left( e^{-1} \dot{X}^2 - em^2 \right) , \quad (1.8)$$

where we've used the notation  $\dot{X}^2 = \dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu}$ . For the rest of these lectures, terms like  $X^2$  will always mean an implicit contraction with the spacetime Minkowski metric.

This form of the action makes it look as if we have coupled the worldline theory to 1d gravity, with the field  $e(\tau)$  acting as an einbein (in the sense of vierbeins that are introduced in general relativity). To see this, note that we could change notation and write this action in the more suggestive form

$$S = -\frac{1}{2} \int d\tau \sqrt{-g_{\tau\tau}} \left( g^{\tau\tau} \dot{X}^2 + m^2 \right) . \quad (1.9)$$

where  $g_{\tau\tau} = (g^{\tau\tau})^{-1}$  is the metric on the worldline and  $e = \sqrt{-g_{\tau\tau}}$

Although our action appears to have one more degree of freedom,  $e$ , it can be easily checked that it has the same equations of motion as (1.2). The reason for this is that  $e$  is completely fixed by its equation of motion,  $\dot{X}^2 + e^2 m^2 = 0$ . Substituting this into the action (1.8) recovers (1.2)

The action (1.8) has a couple of advantages over (1.2). Firstly, it works for massless particles with  $m = 0$ . Secondly, the absence of the annoying square root means that it's easier to quantize in a path integral framework.

The action (1.8) retains invariance under reparameterizations which are now written in a form that looks more like general relativity. For transformations parameterized by an infinitesimal  $\eta$ , we have

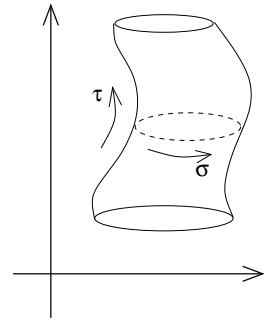
$$\tau \rightarrow \tilde{\tau} = \tau - \eta(\tau) , \quad \delta e = \frac{d}{d\tau}(\eta(\tau)e) , \quad \delta X^\mu = \frac{dX^\mu}{d\tau} \eta(\tau) \quad (1.10)$$

The einbein  $e$  transforms as a density on the worldline, while each of the coordinates  $X^\mu$  transforms as a worldline scalar.

## 1.2 The Nambu-Goto Action

A particle sweeps out a worldline in Minkowski space. A string sweeps out a *worldsheet*. We'll parameterize this worldsheet by one timelike coordinate  $\tau$ , and one spacelike coordinate  $\sigma$ . In this section we'll focus on closed strings and take  $\sigma$  to be periodic, with range

$$\sigma \in [0, 2\pi) . \quad (1.11)$$



**Figure 5:**

We will sometimes package the two worldsheet coordinates together as  $\sigma^\alpha = (\tau, \sigma)$ ,  $\alpha = 0, 1$ . Then the string sweeps out a surface in spacetime which defines a map from the worldsheet to Minkowski space,  $X^\mu(\sigma, \tau)$  with  $\mu = 0, \dots, D - 1$ . For closed strings, we require

$$X^\mu(\sigma, \tau) = X^\mu(\sigma + 2\pi, \tau) .$$

In this context, spacetime is sometimes referred to as the *target space* to distinguish it from the worldsheet.

We need an action that describes the dynamics of this string. The key property that we will ask for is that nothing depends on the coordinates  $\sigma^\alpha$  that we choose on the worldsheet. In other words, the string action should be reparameterization invariant. What kind of action does the trick? Well, for the point particle the action was proportional to the length of the worldline. The obvious generalization is that the action for the string should be proportional to the area,  $A$ , of the worldsheet. This is certainly a property that is characteristic of the worldsheet itself, rather than any choice of parameterization.

How do we find the area  $A$  in terms of the coordinates  $X^\mu(\sigma, \tau)$ ? The worldsheet is a curved surface embedded in spacetime. The induced metric,  $\gamma_{\alpha\beta}$ , on this surface is the pull-back of the flat metric on Minkowski space,

$$\gamma_{\alpha\beta} = \frac{\partial X^\mu}{\partial \sigma^\alpha} \frac{\partial X^\nu}{\partial \sigma^\beta} \eta_{\mu\nu} . \quad (1.12)$$

Then the action which is proportional to the area of the worldsheet is given by,

$$S = -T \int d^2\sigma \sqrt{-\det \gamma} . \quad (1.13)$$

Here  $T$  is a constant of proportionality. We will see shortly that it is the *tension* of the string, meaning the mass per unit length.

We can write this action a little more explicitly. The pull-back of the metric is given by,

$$\gamma_{\alpha\beta} = \begin{pmatrix} \dot{X}^2 & \dot{X} \cdot X' \\ \dot{X} \cdot X' & X'^2 \end{pmatrix} .$$

where  $\dot{X}^\mu = \partial X^\mu / \partial \tau$  and  $X^{\mu'} = \partial X^\mu / \partial \sigma$ . The action then takes the form,

$$S = -T \int d^2\sigma \sqrt{-(\dot{X})^2 (X')^2 + (\dot{X} \cdot X')^2} . \quad (1.14)$$

This is the *Nambu-Goto* action for a relativistic string.

### Action = Area: A Check

If you're unfamiliar with differential geometry, the argument about the pull-back of the metric may be a bit slick. Thankfully, there's a more pedestrian way to see that the action (1.14) is equal to the area swept out by the worldsheet. It's slightly simpler to make this argument for a surface embedded in Euclidean space rather than Minkowski space. We choose some parameterization of the sheet in terms of  $\tau$  and  $\sigma$ , as drawn in the figure, and we write the coordinates of Euclidean space as  $\vec{X}(\sigma, \tau)$ . We'll compute the area of the infinitesimal shaded region. The vectors tangent to the boundary are,

$$\vec{dl}_1 = \frac{\partial \vec{X}}{\partial \sigma} , \quad \vec{dl}_2 = \frac{\partial \vec{X}}{\partial \tau} .$$

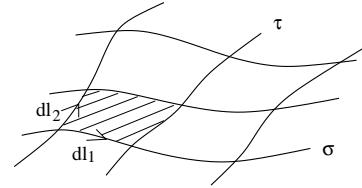
If the angle between these two vectors is  $\theta$ , then the area is then given by

$$ds^2 = |\vec{dl}_1| |\vec{dl}_2| \sin \theta = \sqrt{dl_1^2 dl_2^2 (1 - \cos^2 \theta)} = \sqrt{dl_1^2 dl_2^2 - (\vec{dl}_1 \cdot \vec{dl}_2)^2} \quad (1.15)$$

which indeed takes the form of the integrand of (1.14).

### Tension and Dimension

Let's now see that  $T$  has the physical interpretation of tension. We write Minkowski coordinates as  $X^\mu = (t, \vec{x})$ . We work in a gauge with  $X^0 \equiv t = R\tau$ , where  $R$  is a constant that is needed to balance up dimensions (see below) and will drop out at the end of the argument. Consider a snapshot of a string configuration at a time when



**Figure 6:**

$d\vec{x}/d\tau = 0$  so that the instantaneous kinetic energy vanishes. Evaluating the action for a time  $dt$  gives

$$S = -T \int d\tau d\sigma R \sqrt{(d\vec{x}/d\sigma)^2} = -T \int dt \text{ (spatial length of string)} . \quad (1.16)$$

But, when the kinetic energy vanishes, the action is proportional to the time integral of the potential energy,

$$\text{potential energy} = T \times \text{ (spatial length of string)} .$$

So  $T$  is indeed the energy per unit length as claimed. We learn that the string acts rather like an elastic band and its energy increases linearly with length. (This is different from the elastic bands you're used to which obey Hooke's law where energy increased quadratically with length). To minimize its potential energy, the string will want to shrink to zero size. We'll see that when we include quantum effects this can't happen because of the usual zero point energies.

There is a slightly annoying way of writing the tension that has its origin in ancient history, but is commonly used today

$$T = \frac{1}{2\pi\alpha'} \quad (1.17)$$

where  $\alpha'$  is pronounced "alpha-prime". In the language of our ancestors,  $\alpha'$  is referred to as the "universal Regge slope". We'll explain why later in this course.

At this point, it's worth pointing out some conventions that we have, until now, left implicit. The spacetime coordinates have dimension  $[X] = -1$ . In contrast, the worldsheet coordinates are taken to be dimensionless,  $[\sigma] = 0$ . (This can be seen in our identification  $\sigma \equiv \sigma + 2\pi$ ). The tension is equal to the mass per unit length and has dimension  $[T] = 2$ . Obviously this means that  $[\alpha'] = -2$ . We can therefore associate a length scale,  $l_s$ , by

$$\alpha' = l_s^2 \quad (1.18)$$

The *string scale*  $l_s$  is the natural length that appears in string theory. In fact, in a certain sense (that we will make more precise below) this length scale is the only parameter of the theory.

## Actual Strings vs. Fundamental Strings

There are several situations in Nature where string-like objects arise. Prime examples include magnetic flux tubes in superconductors and chromo-electric flux tubes in QCD. Cosmic strings, a popular speculation in cosmology, are similar objects, stretched across the sky. In each of these situations, there are typically two length scales associated to the string: the tension,  $T$  and the width of the string,  $L$ . For all these objects, the dynamics is governed by the Nambu-Goto action as long as the curvature of the string is much greater than  $L$ . (In the case of superconductors, one should work with a suitable non-relativistic version of the Nambu-Goto action).

However, in each of these other cases, the Nambu-Goto action is not the end of the story. There will typically be additional terms in the action that depend on the width of the string. The form of these terms is not universal, but often includes a *rigidity* piece of form  $L \int K^2$ , where  $K$  is the extrinsic curvature of the worldsheet. Other terms could be added to describe fluctuations in the width of the string.

The string scale,  $l_s$ , or equivalently the tension,  $T$ , depends on the kind of string that we're considering. For example, if we're interested in QCD flux tubes then we would take

$$T \sim (1 \text{ GeV})^2 \quad (1.19)$$

In this course we will consider *fundamental strings* which have zero width. What this means in practice is that we take the Nambu-Goto action as the complete description for all configurations of the string. These strings will have relevance to quantum gravity and the tension of the string is taken to be much larger, typically an order of magnitude or so below the Planck scale.

$$T \lesssim M_{pl}^2 = (10^{18} \text{ GeV})^2 \quad (1.20)$$

However, I should point out that when we try to view string theory as a fundamental theory of quantum gravity, we don't really know what value  $T$  should take. As we will see later in this course, it depends on many other aspects, most notably the string coupling and the volume of the extra dimensions.

### 1.2.1 Symmetries of the Nambu-Goto Action

The Nambu-Goto action has two types of symmetry, each of a different nature.

- Poincaré invariance of the spacetime (1.6). This is a global symmetry from the perspective of the worldsheet, meaning that the parameters  $\Lambda^\mu_\nu$  and  $c^\mu$  which label

the symmetry transformation are constants and do not depend on worldsheet coordinates  $\sigma^\alpha$ .

- Reparameterization invariance,  $\sigma^\alpha \rightarrow \tilde{\sigma}^\alpha(\sigma)$ . As for the point particle, this is a gauge symmetry. It reflects the fact that we have a redundancy in our description because the worldsheet coordinates  $\sigma^\alpha$  have no physical meaning.

### 1.2.2 Equations of Motion

To derive the equations of motion for the Nambu-Goto string, we first introduce the momenta which we call  $\Pi$  because there will be countless other quantities that we want to call  $p$  later,

$$\begin{aligned}\Pi_\mu^\tau &= \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = -T \frac{(\dot{X} \cdot X') X'_\mu - (X'^2) \dot{X}_\mu}{\sqrt{(\dot{X} \cdot X')^2 - \dot{X}^2 X'^2}} . \\ \Pi_\mu^\sigma &= \frac{\partial \mathcal{L}}{\partial X'^\mu} = -T \frac{(\dot{X} \cdot X') \dot{X}_\mu - (\dot{X}^2) X'_\mu}{\sqrt{(\dot{X} \cdot X')^2 - \dot{X}^2 X'^2}} .\end{aligned}$$

The equations of motion are then given by,

$$\frac{\partial \Pi_\mu^\tau}{\partial \tau} + \frac{\partial \Pi_\mu^\sigma}{\partial \sigma} = 0$$

These look like nasty, non-linear equations. In fact, there's a slightly nicer way to write these equations, starting from the earlier action (1.13). Recall that the variation of a determinant is  $\delta \sqrt{-\gamma} = \frac{1}{2} \sqrt{-\gamma} \gamma^{\alpha\beta} \delta \gamma_{\alpha\beta}$ . Using the definition of the pull-back metric  $\gamma_{\alpha\beta}$ , this gives rise to the equations of motion

$$\partial_\alpha (\sqrt{-\det \gamma} \gamma^{\alpha\beta} \partial_\beta X^\mu) = 0 , \quad (1.21)$$

Although this notation makes the equations look a little nicer, we're kidding ourselves. Written in terms of  $X^\mu$ , they are still the same equations. Still nasty.

### 1.3 The Polyakov Action

The square-root in the Nambu-Goto action means that it's rather difficult to quantize using path integral techniques. However, there is another form of the string action which is classically equivalent to the Nambu-Goto action. It eliminates the square root at the expense of introducing another field,

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-g} g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \quad (1.22)$$

where  $g \equiv \det g$ . This is the *Polyakov* action. (Polyakov didn't discover the action, but he understood how to work with it in the path integral, and for this reason it carries his name. The path integral treatment of this action will be the subject of Chapter 5).

The new field is  $g_{\alpha\beta}$ . It is a dynamical metric on the worldsheet. From the perspective of the worldsheet, the Polyakov action is a bunch of scalar fields  $X$  coupled to 2d gravity.

The equation of motion for  $X^\mu$  is

$$\partial_\alpha(\sqrt{-g}g^{\alpha\beta}\partial_\beta X^\mu) = 0 , \quad (1.23)$$

which coincides with the equation of motion (1.21) from the Nambu-Goto action, except that  $g_{\alpha\beta}$  is now an independent variable which is fixed by its own equation of motion. To determine this, we vary the action (remembering again that  $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\alpha\beta}\delta g^{\alpha\beta} = +\frac{1}{2}\sqrt{-g}g^{\alpha\beta}\delta g_{\alpha\beta}$ ),

$$\delta S = -\frac{T}{2} \int d^2\sigma \delta g^{\alpha\beta} (\sqrt{-g} \partial_\alpha X^\mu \partial_\beta X^\nu - \frac{1}{2}\sqrt{-g} g_{\alpha\beta} g^{\rho\sigma} \partial_\rho X^\mu \partial_\sigma X^\nu) \eta_{\mu\nu} = 0 . \quad (1.24)$$

The worldsheet metric is therefore given by,

$$g_{\alpha\beta} = 2f(\sigma) \partial_\alpha X \cdot \partial_\beta X , \quad (1.25)$$

where the function  $f(\sigma)$  is given by,

$$f^{-1} = g^{\rho\sigma} \partial_\rho X \cdot \partial_\sigma X$$

A comment on the potentially ambiguous notation: here, and below, any function  $f(\sigma)$  is always short-hand for  $f(\sigma, \tau)$ : it in no way implies that  $f$  depends only on the spatial worldsheet coordinate.

We see that  $g_{\alpha\beta}$  isn't quite the same as the pull-back metric  $\gamma_{\alpha\beta}$  defined in equation (1.12); the two differ by the conformal factor  $f$ . However, this doesn't matter because, rather remarkably,  $f$  drops out of the equation of motion (1.23). This is because the  $\sqrt{-g}$  term scales as  $f$ , while the inverse metric  $g^{\alpha\beta}$  scales as  $f^{-1}$  and the two pieces cancel. We therefore see that Nambu-Goto and the Polyakov actions result in the same equation of motion for  $X$ .

In fact, we can see more directly that the Nambu-Goto and Polyakov actions coincide. We may replace  $g_{\alpha\beta}$  in the Polyakov action (1.22) with its equation of motion  $g_{\alpha\beta} = 2f\gamma_{\alpha\beta}$ . The factor of  $f$  also drops out of the action for the same reason that it dropped out of the equation of motion. In this manner, we recover the Nambu-Goto action (1.13).

### 1.3.1 Symmetries of the Polyakov Action

The fact that the presence of the factor  $f(\sigma, \tau)$  in (1.25) didn't actually affect the equations of motion for  $X^\mu$  reflects the existence of an extra symmetry which the Polyakov action enjoys. Let's look more closely at this. Firstly, the Polyakov action still has the two symmetries of the Nambu-Goto action,

- Poincaré invariance. This is a global symmetry on the worldsheet.

$$X^\mu \rightarrow \Lambda^\mu_\nu X^\nu + c^\mu .$$

- Reparameterization invariance, also known as diffeomorphisms. This is a gauge symmetry on the worldsheet. We may redefine the worldsheet coordinates as  $\sigma^\alpha \rightarrow \tilde{\sigma}^\alpha(\sigma)$ . The fields  $X^\mu$  transform as worldsheet scalars, while  $g_{\alpha\beta}$  transforms in the manner appropriate for a 2d metric.

$$\begin{aligned} X^\mu(\sigma) &\rightarrow \tilde{X}^\mu(\tilde{\sigma}) = X^\mu(\sigma) \\ g_{\alpha\beta}(\sigma) &\rightarrow \tilde{g}_{\alpha\beta}(\tilde{\sigma}) = \frac{\partial \sigma^\gamma}{\partial \tilde{\sigma}^\alpha} \frac{\partial \sigma^\delta}{\partial \tilde{\sigma}^\beta} g_{\gamma\delta}(\sigma) \end{aligned}$$

It will sometimes be useful to work infinitesimally. If we make the coordinate change  $\sigma^\alpha \rightarrow \tilde{\sigma}^\alpha = \sigma^\alpha - \eta^\alpha(\sigma)$ , for some small  $\eta$ . The transformations of the fields then become,

$$\begin{aligned} \delta X^\mu(\sigma) &= \eta^\alpha \partial_\alpha X^\mu \\ \delta g_{\alpha\beta}(\sigma) &= \nabla_\alpha \eta_\beta + \nabla_\beta \eta_\alpha \end{aligned}$$

where the covariant derivative is defined by  $\nabla_\alpha \eta_\beta = \partial_\alpha \eta_\beta - \Gamma_{\alpha\beta}^\sigma \eta_\sigma$  with the Levi-Civita connection associated to the worldsheet metric given by the usual expression,

$$\Gamma_{\alpha\beta}^\sigma = \frac{1}{2} g^{\sigma\rho} (\partial_\alpha g_{\beta\rho} + \partial_\beta g_{\rho\alpha} - \partial_\rho g_{\alpha\beta})$$

Together with these familiar symmetries, there is also a new symmetry which is novel to the Polyakov action. It is called *Weyl invariance*.

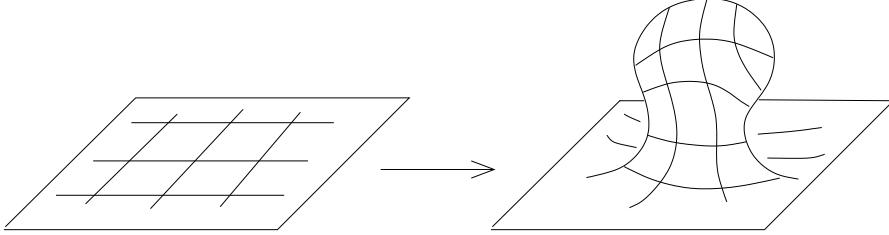
- Weyl Invariance. Under this symmetry,  $X^\mu(\sigma) \rightarrow X^\mu(\sigma)$ , while the metric changes as

$$g_{\alpha\beta}(\sigma) \rightarrow \Omega^2(\sigma) g_{\alpha\beta}(\sigma) . \quad (1.26)$$

Or, infinitesimally, we can write  $\Omega^2(\sigma) = e^{2\phi(\sigma)}$  for small  $\phi$  so that

$$\delta g_{\alpha\beta}(\sigma) = 2\phi(\sigma) g_{\alpha\beta}(\sigma) .$$

It is simple to see that the Polyakov action is invariant under this transformation: the factor of  $\Omega^2$  drops out just as the factor of  $f$  did in equation (1.25), canceling between  $\sqrt{-g}$  and the inverse metric  $g^{\alpha\beta}$ . This is a gauge symmetry of the string, as seen by the fact that the parameter  $\Omega$  depends on the worldsheet coordinates  $\sigma$ . This means that two metrics which are related by a Weyl transformation (1.26) are to be considered as the same physical state.



**Figure 7:** An example of a Weyl transformation

How should we think of Weyl invariance? It is not a coordinate change. Instead it is the invariance of the theory under a local change of scale which preserves the angles between all lines. For example the two worldsheet metrics shown in the figure are viewed by the Polyakov string as equivalent. This is rather surprising! And, as you might imagine, theories with this property are extremely rare. It should be clear from the discussion above that the property of Weyl invariance is special to two dimensions, for only there does the scaling factor coming from the determinant  $\sqrt{-g}$  cancel that coming from the inverse metric. But even in two dimensions, if we wish to keep Weyl invariance then we are strictly limited in the kind of interactions that can be added to the action. For example, we would not be allowed a potential term for the worldsheet scalars of the form,

$$\int d^2\sigma \sqrt{-g} V(X) .$$

These break Weyl invariance. Nor can we add a worldsheet cosmological constant term,

$$\mu \int d^2\sigma \sqrt{-g} .$$

This too breaks Weyl invariance. We will see later in this course that the requirement of Weyl invariance becomes even more stringent in the quantum theory. We will also see what kind of interactions terms can be added to the worldsheet. Indeed, much of this course can be thought of as the study of theories with Weyl invariance.

### 1.3.2 Fixing a Gauge

As we have seen, the equation of motion (1.23) looks pretty nasty. However, we can use the redundancy inherent in the gauge symmetry to choose coordinates in which they simplify. Let's think about what we can do with the gauge symmetry.

Firstly, we have two reparameterizations to play with. The worldsheet metric has three independent components. This means that we expect to be able to set any two of the metric components to a value of our choosing. We will choose to make the metric locally conformally flat, meaning

$$g_{\alpha\beta} = e^{2\phi} \eta_{\alpha\beta} , \quad (1.27)$$

where  $\phi(\sigma, \tau)$  is some function on the worldsheet. You can check that this is possible by writing down the change of the metric under a coordinate transformation and seeing that the differential equations which result from the condition (1.27) have solutions, at least locally. Choosing a metric of the form (1.27) is known as *conformal gauge*.

We have only used reparameterization invariance to get to the metric (1.27). We still have Weyl transformations to play with. Clearly, we can use these to remove the last independent component of the metric and set  $\phi = 0$  such that,

$$g_{\alpha\beta} = \eta_{\alpha\beta} . \quad (1.28)$$

We end up with the flat metric on the worldsheet in Minkowski coordinates.

### A Diversion: How to make a metric flat

The fact that we can use Weyl invariance to make any two-dimensional metric flat is an important result. Let's take a quick diversion from our main discussion to see a different proof that isn't tied to the choice of Minkowski coordinates on the worldsheet. We'll work in 2d Euclidean space to avoid annoying minus signs. Consider two metrics related by a Weyl transformation,  $g'_{\alpha\beta} = e^{2\phi} g_{\alpha\beta}$ . One can check that the Ricci scalars of the two metrics are related by,

$$\sqrt{g'} R' = \sqrt{g} (R - 2\nabla^2 \phi) . \quad (1.29)$$

We can therefore pick a  $\phi$  such that the new metric has vanishing Ricci scalar,  $R' = 0$ , simply by solving this differential equation for  $\phi$ . However, in two dimensions (but not in higher dimensions) a vanishing Ricci scalar implies a flat metric. The reason is simply that there aren't too many indices to play with. In particular, symmetry of the Riemann tensor in two dimensions means that it must take the form,

$$R_{\alpha\beta\gamma\delta} = \frac{R}{2} (g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}) .$$

So  $R' = 0$  is enough to ensure that  $R'_{\alpha\beta\gamma\delta} = 0$ , which means that the manifold is flat. In equation (1.28), we've further used reparameterization invariance to pick coordinates in which the flat metric is the Minkowski metric.

### The equations of motion and the stress-energy tensor

With the choice of the flat metric (1.28), the Polyakov action simplifies tremendously and becomes the theory of  $D$  free scalar fields. (In fact, this simplification happens in any conformal gauge).

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \partial_\alpha X \cdot \partial^\alpha X , \quad (1.30)$$

and the equations of motion for  $X^\mu$  reduce to the free wave equation,

$$\partial_\alpha \partial^\alpha X^\mu = 0 . \quad (1.31)$$

Now that looks too good to be true! Are the horrible equations (1.23) really equivalent to a free wave equation? Well, not quite. There is something that we've forgotten: we picked a choice of gauge for the metric  $g_{\alpha\beta}$ . But we must still make sure that the equation of motion for  $g_{\alpha\beta}$  is satisfied. In fact, the variation of the action with respect to the metric gives rise to a rather special quantity: it is the stress-energy tensor,  $T_{\alpha\beta}$ . With a particular choice of normalization convention, we define the stress-energy tensor to be

$$T_{\alpha\beta} = -\frac{2}{T} \frac{1}{\sqrt{-g}} \frac{\partial S}{\partial g^{\alpha\beta}} .$$

We varied the Polyakov action with respect to  $g_{\alpha\beta}$  in (1.24). When we set  $g_{\alpha\beta} = \eta_{\alpha\beta}$  we get

$$T_{\alpha\beta} = \partial_\alpha X \cdot \partial_\beta X - \frac{1}{2} \eta_{\alpha\beta} \eta^{\rho\sigma} \partial_\rho X \cdot \partial_\sigma X . \quad (1.32)$$

The equation of motion associated to the metric  $g_{\alpha\beta}$  is simply  $T_{\alpha\beta} = 0$ . Or, more explicitly,

$$\begin{aligned} T_{01} &= \dot{X} \cdot X' = 0 \\ T_{00} = T_{11} &= \frac{1}{2} (\dot{X}^2 + X'^2) = 0 . \end{aligned} \quad (1.33)$$

We therefore learn that the equations of motion of the string are the free wave equations (1.31) subject to the two constraints (1.33) arising from the equation of motion  $T_{\alpha\beta} = 0$ .

## Getting a feel for the constraints

Let's try to get some intuition for these constraints. There is a simple meaning of the first constraint in (1.33): we must choose our parameterization such that lines of constant  $\sigma$  are perpendicular to the lines of constant  $\tau$ , as shown in the figure.

But we can do better. To gain more physical insight, we need to make use of the fact that we haven't quite exhausted our gauge symmetry. We will discuss this more in Section 2.2, but for now one can check that there is enough remnant gauge symmetry to allow us to go to static gauge,

$$X^0 \equiv t = R\tau ,$$

so that  $(X^0)' = 0$  and  $\dot{X}^0 = R$ , where  $R$  is a constant that is needed on dimensional grounds. The interpretation of this constant will become clear shortly. Then, writing  $X^\mu = (t, \vec{x})$ , the equation of motion for spatial components is the free wave equation,

$$\ddot{\vec{x}} - \vec{x}'' = 0$$

while the constraints become

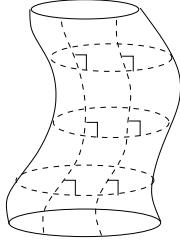
$$\begin{aligned} \dot{\vec{x}} \cdot \vec{x}' &= 0 \\ \dot{\vec{x}}^2 + \vec{x}'^2 &= R^2 \end{aligned} \tag{1.34}$$

The first constraint tells us that the motion of the string must be perpendicular to the string itself. In other words, the physical modes of the string are transverse oscillations. There is no longitudinal mode. We'll also see this again in Section 2.2.

From the second constraint, we can understand the meaning of the constant  $R$ : it is related to the length of the string when  $\dot{\vec{x}} = 0$ ,

$$\int d\sigma \sqrt{(d\vec{x}/d\sigma)^2} = 2\pi R .$$

Of course, if we have a stretched string with  $\dot{\vec{x}} = 0$  at one moment of time, then it won't stay like that for long. It will contract under its own tension. As this happens, the second constraint equation relates the length of the string to the instantaneous velocity of the string.



**Figure 8:**

## 1.4 Mode Expansions

Let's look at the equations of motion and constraints more closely. The equations of motion (1.31) are easily solved. We introduce lightcone coordinates on the worldsheet,

$$\sigma^\pm = \tau \pm \sigma ,$$

in terms of which the equations of motion simply read

$$\partial_+ \partial_- X^\mu = 0$$

The most general solution is,

$$X^\mu(\sigma, \tau) = X_L^\mu(\sigma^+) + X_R^\mu(\sigma^-)$$

for arbitrary functions  $X_L^\mu$  and  $X_R^\mu$ . These describe left-moving and right-moving waves respectively. Of course the solution must still obey both the constraints (1.33) as well as the periodicity condition,

$$X^\mu(\sigma, \tau) = X^\mu(\sigma + 2\pi, \tau) . \quad (1.35)$$

The most general, periodic solution can be expanded in Fourier modes,

$$\begin{aligned} X_L^\mu(\sigma^+) &= \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \tilde{\alpha}_n^\mu e^{-in\sigma^+} , \\ X_R^\mu(\sigma^-) &= \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu \sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\sigma^-} . \end{aligned} \quad (1.36)$$

This mode expansion will be very important when we come to the quantum theory. Let's make a few simple comments here.

- Various normalizations in this expression, such as the  $\alpha'$  and factor of  $1/n$  have been chosen for later convenience.
- $X_L$  and  $X_R$  do not individually satisfy the periodicity condition (1.35) due to the terms linear in  $\sigma^\pm$ . However, the sum of them is invariant under  $\sigma \rightarrow \sigma + 2\pi$  as required.
- The variables  $x^\mu$  and  $p^\mu$  are the position and momentum of the center of mass of the string. This can be checked, for example, by studying the Noether currents arising from the spacetime translation symmetry  $X^\mu \rightarrow X^\mu + c^\mu$ . One finds that the conserved charge is indeed  $p^\mu$ .
- Reality of  $X^\mu$  requires that the coefficients of the Fourier modes,  $\alpha_n^\mu$  and  $\tilde{\alpha}_n^\mu$ , obey

$$\alpha_n^\mu = (\alpha_{-n}^\mu)^* , \quad \tilde{\alpha}_n^\mu = (\tilde{\alpha}_{-n}^\mu)^* . \quad (1.37)$$

### 1.4.1 The Constraints Revisited

We still have to impose the two constraints (1.33). In the worldsheet lightcone coordinates  $\sigma^\pm$ , these become,

$$(\partial_+ X)^2 = (\partial_- X)^2 = 0 . \quad (1.38)$$

These equations give constraints on the momenta  $p^\mu$  and the Fourier modes  $\alpha_n^\mu$  and  $\tilde{\alpha}_n^\mu$ . To see what these are, let's look at

$$\begin{aligned} \partial_- X^\mu &= \partial_- X_R^\mu = \frac{\alpha'}{2} p^\mu + \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \alpha_n^\mu e^{-in\sigma^-} \\ &= \sqrt{\frac{\alpha'}{2}} \sum_n \alpha_n^\mu e^{-in\sigma^-} \end{aligned}$$

where in the second line the sum is over all  $n \in \mathbf{Z}$ , and we have defined  $\alpha_0^\mu$  to be

$$\alpha_0^\mu \equiv \sqrt{\frac{\alpha'}{2}} p^\mu .$$

The constraint (1.38) can then be written as

$$\begin{aligned} (\partial_- X)^2 &= \frac{\alpha'}{2} \sum_{m,p} \alpha_m \cdot \alpha_p e^{-i(m+p)\sigma^-} \\ &= \frac{\alpha'}{2} \sum_{m,n} \alpha_m \cdot \alpha_{n-m} e^{-in\sigma^-} \\ &\equiv \alpha' \sum_n L_n e^{-in\sigma^-} = 0 . \end{aligned}$$

where we have defined the sum of oscillator modes,

$$L_n = \frac{1}{2} \sum_m \alpha_{n-m} \cdot \alpha_m . \quad (1.39)$$

We can also do the same for the left-moving modes, where we again define an analogous sum of operator modes,

$$\tilde{L}_n = \frac{1}{2} \sum_m \tilde{\alpha}_{n-m} \cdot \tilde{\alpha}_m . \quad (1.40)$$

with the zero mode defined to be,

$$\tilde{\alpha}_0^\mu \equiv \sqrt{\frac{\alpha'}{2}} p^\mu .$$

The fact that  $\tilde{\alpha}_0^\mu = \alpha_0^\mu$  looks innocuous but is a key point to remember when we come to quantize the string. The  $L_n$  and  $\tilde{L}_n$  are the Fourier modes of the constraints. Any classical solution of the string of the form (1.36) must further obey the infinite number of constraints,

$$L_n = \tilde{L}_n = 0 \quad n \in \mathbf{Z} .$$

We'll meet these objects  $L_n$  and  $\tilde{L}_n$  again in a more general context when we come to discuss conformal field theory.

The constraints arising from  $L_0$  and  $\tilde{L}_0$  have a rather special interpretation. This is because they include the square of the spacetime momentum  $p^\mu$ . But, the square of the spacetime momentum is an important quantity in Minkowski space: it is the square of the rest mass of a particle,

$$p_\mu p^\mu = -M^2 .$$

So the  $L_0$  and  $\tilde{L}_0$  constraints tell us the effective mass of a string in terms of the excited oscillator modes, namely

$$M^2 = \frac{4}{\alpha'} \sum_{n>0} \alpha_n \cdot \alpha_{-n} = \frac{4}{\alpha'} \sum_{n>0} \tilde{\alpha}_n \cdot \tilde{\alpha}_{-n} \quad (1.41)$$

Because both  $\alpha_0^\mu$  and  $\tilde{\alpha}_0^\mu$  are equal to  $\sqrt{\alpha'/2} p^\mu$ , we have two expressions for the invariant mass: one in terms of right-moving oscillators  $\alpha_n^\mu$  and one in terms of left-moving oscillators  $\tilde{\alpha}_n^\mu$ . And these two terms must be equal to each other. This is known as *level matching*. It will play an important role in the next section where we turn to the quantum theory.